1. Let $f : \mathbb{N} \to \mathbb{R}$ be a function. Let $S_N = \sum_{n=1}^{N} e^{2\pi i f(n)}$. Show that for any $H \le N$, one has

$$|S_N|^2 \le C \frac{N}{H} \sum_{h=0}^{H-1} \left| \sum_{n=1}^{N-h} e^{2\pi i (f(n+h) - f(n))} \right|,$$

for some constant $C > 0$ independent of H, N and f.

Ch. 4 Problem 2 :

Proof :

Let

$$a_n = \begin{cases} e^{2\pi i f(n)} & 1 \le n \le N \\ 0 & \text{otherwise} \end{cases}$$

$$H S_N = H \sum_{n=1}^{N} a_n$$

$$= \sum_{k=1}^{H} \sum_{n=-H+1}^{N-1} a_{n+k}$$

$$= \sum_{n=-H+1}^{N-1} \sum_{k=1}^{H} a_{n+k}$$

By Cauchy - Schwarz inequality,

$$|HS_N|^2 \le \left(\sum_{n=-H+1}^{N-1} 1^2 \right) \left(\sum_{n=-H+1}^{N-1} \left| \sum_{k=1}^{H} a_{n+k} \right|^2 \right)$$

$$= (N+H-1)\cdot\left(\sum_{h=-H+1}^{N-1}\left(\sum_{i=1}^{H}\sum_{j=1}^{H} a_{n+i}\overline{a_{n+j}}\right)\right)$$

From now on, we consider

$$\sum_{n=-\infty}^{\infty}\sum_{i=1}^{H}\sum_{j=1}^{H} a_{n+i}\overline{a_{n+j}}$$

$$= \Big(\underbrace{\sum_{i=1}^{H}\sum_{j=1}^{i-1}}_{\mathrm{I}} + \underbrace{\sum_{i=1}^{H}\sum_{j=i}^{H}}_{\mathrm{II}}\Big)\Big(\sum_{n=-\infty}^{\infty} a_{n+i}\overline{a_{n+j}}\Big)$$

(I) : $1 \le j < i \le H$

(II) : $1 \le i \le j \le H$

(I) : Let $1 \le h = i-j \le H-1$

For each h, note that $1 \le j < i \le H$,

$\Rightarrow 1 \le j$ & $i = h+j \le H$

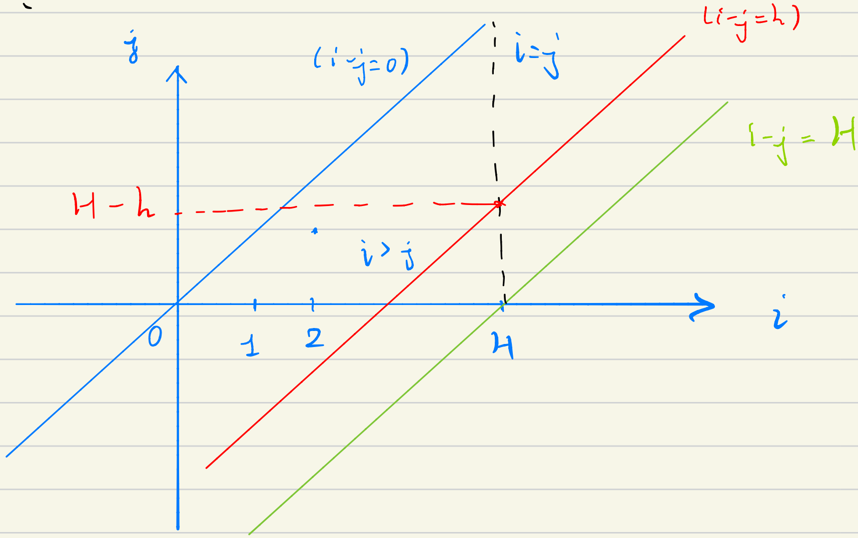
i.e. $1 \le j \le H-h$

$$\therefore \quad (\mathrm{I}) = \sum_{i=1}^{H} \sum_{j=1}^{i-1} \sum_{n=-\infty}^{\infty} a_{n+i}\overline{a_{n+j}}$$

$$= \sum_{h=1}^{H-1} \sum_{j=1}^{H-h} \sum_{n=-\infty}^{\infty} a_{n+h}\overline{a_n}$$

$$= \sum_{h=1}^{H-1} (H-h) \sum_{n} a_{n+h}\overline{a_n}$$

Picture :



We are summing up the lattices through the lines $i-j=h$ $(1 \le h \le H-1)$

And for each fixed line $i-j=h$, we restrict on $1 \le j \le H-h$

(II) : $\quad 1 \le i \le j \le H$

$$\sum_{i=1}^{H} \sum_{j=i}^{H} \sum_{n} a_{n+i} \overline{a_{n+j}}$$

$$= \sum_{h=0}^{H-1} \sum_{i=1}^{H-h} \sum_{n} a_n \overline{a_{n+h}} \qquad (h = j - i)$$

verify yourself

$$= \sum_{h=0}^{H-1} (H-h) \sum_{n} a_n \overline{a_{n+h}}$$

(I) + (II) :

$$\underbrace{H \sum_{n} |a_n|^2}_{h=0} + \sum_{h=1}^{H-1} (H-h) \sum_{n} 2\mathrm{Re}\left(a_n \overline{a_{n+h}}\right)$$

$$\le 2 \sum_{h=0}^{H-1} H \cdot \left| \sum_{n} a_n \overline{a_{n+h}} \right|$$

Recall

$$|HS_N|^2 \leq (N+H-1) \cdot \left(\sum_{h=-H+1}^{N-1} \left(\sum_{i=1}^{H} \sum_{j=1}^{H} a_{n+i} \overline{a_{n+j}} \right) \right)$$

We have

$$|HS_N|^2 \leq 2N \cdot 2H \sum_{h=0}^{H-1} \left| \sum_{n} a_n \overline{a_{n+h}} \right|$$

a_n, a_{n+h} is nonzero only if

$1 \leq n, n+h \leq N$

$$= 4NH \sum_{h=0}^{H-1} \left| \sum_{n=1}^{N-h} a_n \overline{a_{n+h}} \right|$$

$$= 4NH \sum_{h=0}^{H-1} \left| \sum_{n=1}^{N-h} e^{2\pi i (f(n+h) - f(n))} \right|$$

#

2. Show that the sequence $\langle \gamma n^2 \rangle$ is equidistributed in $[0, 1)$ whenever γ is irrational.

Proof: For any $k \in \mathbb{Z} \setminus \{0\}$,

we want to show that

$$\frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k \gamma n^2} \longrightarrow 0 \quad \text{as } N \to \infty$$

Let $f(n) = k\gamma n^2$ and use (a):

$$\left| \sum_{n=1}^{N} e^{2\pi i f(n)} \right|^2 \leq C \cdot \frac{N}{H} \sum_{h=0}^{H-1} \left| \sum_{n=1}^{N-h} e^{2\pi i (f(n+h) - f(n))} \right|$$

$$= C \cdot \frac{N}{H} \sum_{h=0}^{H-1} \left| \sum_{n=1}^{N-h} e^{2\pi i k\gamma (2nh)} \cdot \underbrace{e^{2\pi i k\gamma h^2}}_{\text{modulus} = 1} \right|$$

$$= C \cdot \frac{N}{H} \sum_{h=0}^{H-1} \left| \sum_{n=1}^{N-h} e^{2\pi i (2hk) n\gamma} \right|$$

if $h \neq 0$

we may apply the known result:

$\langle n\gamma \rangle$ is equidistributed in $[0,1)$

i.e. $\frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k n\gamma} \longrightarrow 0$ as $N \to \infty$ for $k \neq 0$

$$= C \cdot \frac{N}{H} \left| \sum_{n=1}^{N} 1 \right| + C \cdot \frac{N}{H} \sum_{h=1}^{H-1} \left| \sum_{n=1}^{N-h} e^{2\pi i (2hk) n \alpha} \right|$$

$$= C \cdot \frac{N^2}{H} + C \cdot \frac{N}{H} \sum_{h=1}^{H-1} \left| \sum_{n=1}^{N-h} e^{2\pi i (2hk) n \alpha} \right|$$

$$\therefore \quad \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k \alpha n} \right|^2$$

$$\leq \frac{C}{H} + \frac{C}{H} \cdot \frac{1}{N} \sum_{h=1}^{H-1} \left| \sum_{n=1}^{N-h} e^{2\pi i (2hk) n \alpha} \right|$$

Let $\varepsilon > 0$, we may fix H large so that $\frac{C}{H} < \varepsilon/2$.

Note $$\frac{1}{N} \sum_{h=1}^{H-1} \left| \sum_{n=1}^{\boxed{N-h}} e^{2\pi i (2hk) n \alpha} \right|$$

$$\leq \frac{1}{N} \sum_{h=1}^{H-1} \left(\left| \sum_{n=1}^{\boxed{N}} e^{2\pi i (2hk) n \alpha} \right| + h \right)$$

$$= \sum_{h=1}^{H-1} \frac{1}{N} \left| \sum_{n=1}^{N} e^{2\pi i (2hk) n \alpha} \right| + \frac{H(H-1)}{2N}$$

$$\longrightarrow 0 \quad \text{as} \quad N \to \infty$$

3. More generally, show that if $\{x_n\}$ is a sequence in $[0,1)$ such that for all positiver integers h the difference $\langle x_{n+h} - x_n \rangle$ is equidistributed in $[0,1)$, then $\{x_n\}$ is equidistributed in $[0,1)$.

Left as an exercise.

4. Let $P(x) = c_N x^N + \cdots + c_0$ be a polynomial with real coefficients. Suppose at least one of $c_1, \ldots, c_N$ is irrational. Show that the sequence $\langle P(n) \rangle$ is equidistributed in $[0,1)$.

(**Hint:** Argue by induction on the highest degree term which has an irrational coefficient. To prove the base case, consider $P(x) = Q(x) + \gamma x + c$, where γ is irrational and $Q(x)$ is a polynomial with rational coefficients. Find an integer L such that $LQ(x)$ is an integral polynomial. For any $1 \leq n \leq N$, write $n = kL + d$ and show that

$$Q(kL + d) = Q(d) + \text{ an integer.})$$

Proof:

Let $S(m)$ be the statement:

If P is a polynomial with degree $N \geq m$ s.t x^m is the highest degree term with irrational coefficient, then $\langle P(n) \rangle$ is equidistributed.

Say $P(x) = C_N x^N + C_{N-1} x^{N-1} + \cdots + C_0$,

i.e. $C_k \in \mathbb{Q}$ for $k > m$

$C_m \in \mathbb{R} \setminus \mathbb{Q}$

We show that $S(m) \Rightarrow S(m+1)$:

Assume $S(m)$ is true and let

$P(x) = c^N x^N + c^{N-1} x^{N-1} + \dots + c_0$ be a

polynomial with degree $N \geq m+1$, s.t.

x^{m+1} is the highest degree term with

irrational coefficient.

We aim to show that $\langle P(n) \rangle$ is

equidistributed in $[0,1)$:

By Q3, it suffices to check that for

any $h \in \mathbb{N}$, $\langle \langle P(n+h) \rangle - \langle P(n) \rangle \rangle$ is

equidistributed in $[0,1)$.

Check it yourself :

$\langle \langle P(n+h) \rangle - \langle P(n) \rangle \rangle = \langle P(n+h) - P(n) \rangle$

Since h is fixed, we will show that

$P(x+h) - P(x)$ is a polynomial with degree $\geq m$

and x^m is the highest degree term with

irrational coefficient.

The induction hypothesis will conclude that

$\langle P(n+h) - P(n) \rangle_{n \in \mathbb{N}}$ is equidistributed in $[0,1)$.

Back to $P(x+h) - P(x)$:

Let $Q(x) = c_N x^N + \cdots + c_{m+2} x^{m+2}$

$R(x) = c_m x^m + \cdots + c_0$

Then, $P(x) = Q(x) + c_{m+1} x^{m+1} + R(x)$

$$P(x+h) - P(x) = Q(x+h) - Q(x) + R(x+h) - R(x)$$
$$+ c_{m+1} (x+h)^{m+1} - c_{m+1} x^{m+1}$$

$$= Q(x+h) - Q(x) + R(x+h) - R(x) + c_{m+1} \cdot (m+1) h x^m$$
$$+ T_h(x),$$

where $Q(x+h) - Q(x)$ is a polynomial with rational coefficients.

$R(x+h) - R(x)$ is a polynomial with degree $\leq m-1$

$T_h(x)$ is a polynomial with degree $\leq m-1$

This verifies that $P(x+h) - P(x)$ is a polynomial fulfiling the assumption of $S(m)$.

It remains to prove the base case $S(1)$:

Let $P(x) = Q(x) + \gamma x + c$, where

$Q(x) = c_N x^N + \cdots + c_2 x^2$ and

$c_2, c_3, \ldots, c_N \in \mathbb{Q}$

Aim: $\langle P(n) \rangle$ is equidistributed in $[0,1)$.

i.e. For any $k \in \mathbb{Z} \setminus \{0\}$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k P(n)} = 0$$

WLOG, we can assume $k=1$.

Fix $L \in \mathbb{N}$ s.t. $Lc_2, Lc_3, \ldots, Lc_N$ are all integers. For any $N > L$, $\exists$ M, d s.t.

$$N = ML + d_N \quad \text{and} \quad 0 \le d_N \le L-1$$

Now, we inspect the sum

$$\sum_{n=1}^{ML+L} e^{2\pi i P(n)} = \sum_{d=1}^{L} \sum_{k=0}^{M} e^{2\pi i P(kL+d)}$$

$$= \sum_{d=1}^{L} \sum_{k=0}^{M} e^{2\pi i Q(kL+d)} \cdot e^{2\pi i \gamma(kL+d)} \cdot e^{2\pi i c}$$

$$= e^{2\pi i c} \sum_{d=1}^{L} e^{2\pi i \gamma d} \sum_{k=0}^{M} e^{2\pi i Q(kL+d)} e^{2\pi i \gamma kL}$$

Notice that (why?)

$$Q(kL+d) = Q(d) + \text{an integer}$$

$$= e^{2\pi i c} \sum_{d=1}^{L} e^{2\pi i \gamma d} \sum_{k=0}^{M} e^{2\pi i Q(d)} e^{2\pi i \gamma kL}$$

$$= e^{2\pi i c} \sum_{d=1}^{L} e^{2\pi i \gamma d} \cdot e^{2\pi i Q(d)} \cdot \sum_{k=0}^{M} e^{2\pi i \gamma kL}$$

$$\therefore \quad \left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i P(n)}\right| \leq \underbrace{\frac{1}{N}\left|\sum_{n=1}^{ML+L} e^{2\pi i P(n)}\right|} + \underbrace{\frac{1}{N}\left|\sum_{n=N+1}^{ML+L} e^{2\pi i P(n)}\right|}$$

$$\leq \frac{1}{N}\sum_{d=1}^{L}\left|\sum_{k=0}^{M} e^{2\pi i \gamma k L}\right| + \frac{1}{N}\cdot L \quad \left(ML \leq N < ML+L\right)$$

$$\leq \frac{1}{M}\left|\sum_{k=0}^{M} e^{2\pi i \gamma k L}\right| + \frac{1}{M}$$

Since ① $\langle \gamma k L\rangle_k$ is equidistributed in $[0,1)$ (why?),

and ② $M \to \infty$ as $N \to \infty$,

We conclude that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i P(n)} = 0 \qquad \#$$